NUMBER OF BOUND STATES OF THE SCHRÖDINGER OPERATOR OF A SYSTEM OF THREE BOSONS IN AN OPTICAL LATTICE.

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ABSTRACT. We consider the Hamiltonian $\hat{\mathrm{H}}_\mu$ of a system of three identical particles(bosons) on the $d-$ dimensional lattice $\mathbb{Z}^d, d=1,2$ interacting via pairwise zero-range attractive potential $\mu<0$. We describe precise location and structure of the essential spectrum of the Schrödinger operator $H_\mu(K), K\in\mathbb{T}^d$ associated to $\hat{\mathrm{H}}_\mu$ and prove the finiteness of the number of bound states of $H_\mu(K), K\in\mathbb{T}^d$ lying below the bottom of the essential spectrum. Moreover, we show that bound states decay exponentially at infinity and eigenvalues and corresponding bound states of $H_\mu(K), K\in\mathbb{T}^d$ are regular as a function of center of mass quasi-momentum $K\in\mathbb{T}^d$.

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1. INTRODUCTION

Cold atoms loaded in an optical lattice provide a realization of a quantum lattice gas. The periodicity of the potential gives rise to a band structure for the dynamics of the atoms.

The dynamics of the ultracold atoms loaded in the lower band is well described by the Bose-Hubbard hamiltonian [20]; we give in section 2 the corresponding Schödinger operator.

In the continuous case [6],[7, 14] the energy of the center-of-mass motion can by separated out from the total Hamiltonian, i.e., the energy operator can by split into a sum of a center-of-mass motion and a relative kinetic energy. So that the three-particle *bound states* are eigenvectors of the relative kinetic energy operator.

The kinematics of the quantum particles on the lattice is rather exotic. The discrete laplacian *is not translationally invariant* and therefore one cannot separate the motion of the center of mass.

One can rather resort to a Bloch-Floquet decomposition. The three-particle Hilbert space $\mathcal{H}\equiv\ell^2(\mathbb{Z}^d)^3$ is decomposed as direct integral associated to the representation of the discrete group $\mathbb{Z}^d$ by shift operators.

$$\ell^2[(\mathbb{Z}^d)^3]=\int_{\mathbb{T}^d}\oplus\ell^2[(\mathbb{Z}^d)^2]\eta(dK),$$

where $\eta(dp)$ is the (normalized) Haar measure on the torus $\mathbb{T}^d$. Hence the total three-body Hamiltonian H of a system of three particles on $d-$ dimensional lattice $\mathbb{Z}^d, d\geq 1$ interacting via pairwise short range attractive potential V appears to be decomposable

$$\mathrm{H}=\int_{\mathbb{T}^d}\oplus H(K)\eta(dK).$$

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The fiber hamiltonians $H(K)$ depends parametrically on the *quasi momentum* $K \in \mathbb{T}^d \equiv \mathbb{R}^d/(2\pi\mathbb{Z}^d)$. It is the sum of a free part depending on K continuously and an interaction term, both bounded.

Bound states $\psi_{E,K}$ are solution of the Schrödinger equation

$$H(K)\psi_{E,K} = E\psi_{E,K} \qquad \psi_{E,K} \in \ell^2[(\mathbb{Z}^d)^2].$$

The Efimov effect is one of the remarkable results in the spectral analysis of Hamiltonians associated to a system of three-particles moving on the three-dimensional Euclid space: if none of the three two-particle Schrödinger operators (associated to the two-particle subsystems of a three-particle system) has negative eigenvalues, but at least two of them have zero energy resonance, then the three-particle Schrödinger operator has an infinite number of discrete eigenvalues, accumulating at zero [1, 3, 4, 5, 8, 16, 17, 19].

The finiteness of eigenvalues (absence Efimov's effect) have been proved for the Hamiltonian of a system of three particles moving on $d = 1, 2$ dimensional Euclid space $\mathbb{R}^d$ in [18].

We consider the Hamiltonian $\hat{\mathrm{H}}_\mu$ of a system of three identical particles(bosons) on $d-$ dimensional lattice $\mathbb{Z}^d, d = 1, 2,$ interacting via pairwise zero range attractive potential $\mu < 0.$

We prove the finiteness of the number of bound states(absence of Efimov's effect) of the Schrödinger operator $H_\mu(K),\ K \in \mathbb{G} \subseteq \mathbb{T}^d, d = 1, 2,$ where $\mathbb{G} \subseteq \mathbb{T}^d$ is a region, associated to the Hamiltonian $\hat{\mathrm{H}}_\mu$.

We describe a precise location and structure of the essential spectrum of the Schrödinger operator $H_\mu(K), K \in \mathbb{T}^d$. Moreover, we show that bound states decay exponentially at infinity and we establish that the eigenvalues and corresponding bound states of $H_\mu(K), K \in \mathbb{T}^d$ are regular as a function of center of mass quasi-momentum $K \in \mathbb{T}^d$.

In [12] finiteness of the eigenvalues of the discrete Schrödinger operator associated to a system of three-bosons on one dimensional lattice $\mathbb{Z}^1$ has been shown.

Section 1 is an introduction. In Section 2 we introduce the Hamiltonians of systems of two and three-particles in coordinate and momentum representations as bounded self-adjoint operators in the corresponding Hilbert spaces.

In Section 3 we introduce the total quasi-momentum, decompose the energy operators into von Neumann direct integrals, introduce discrete Schrödinger operators $h_\mu(k), k \in \mathbb{T}^d$ and $H_\mu(K), K \in \mathbb{T}^d$, choosing relative coordinate system.

We state the main results in Section 4.

We introduce the *channel operators* and describe the essential spectrum of $H_\mu(K)$ by means of the discrete spectrum of the two particle Schrödinger operators $h_\mu(k), k \in \mathbb{T}^d$ (Theorem 4.2) in section 5.

In Section 6 we prove the finiteness of the number of eigenvalues of the three-particle Schrödinger operator $H_\mu(K), K \in \mathbb{T}^d$ (Theorem 4.3) and finiteness of isolated bands in a system of three particles in an optical lattice.

2. Hamiltonians of three identical particles on a lattices in the coordinate and momentum representations

Let $\mathbb{Z}^d, d = 1, 2$ be the d-dimensional lattice. Let $\ell^2[(\mathbb{Z}^d)^m], d = 1, 2$ be Hilbert space of square-summable functions $\hat{\varphi}$ defined on the Cartesian power of $(\mathbb{Z}^d)^m, d = 1, 2$ and let $\ell^{2,s}[(\mathbb{Z}^d)^m] \subset \ell^2[(\mathbb{Z}^d)^m]$ be the subspace of symmetric functions.

Let Δ be the lattice Laplacian, i.e., the operator which describes the transport of a particle from one site to another site:

$$(\Delta\hat{\psi})(x) = \frac{1}{2}\sum_{|s|=1}[\hat{\psi}(x) - \hat{\psi}(x+s)], \quad \hat{\psi} \in \ell^2(\mathbb{Z}^d).$$

The free Hamiltonian $\hat{\mathrm{h}}_0$ of a system of two identical quantum mechanical particles with mass $m = 1$ on the d-dimensional lattice $\mathbb{Z}^d, d = 1,2$ in the coordinate representation is associated to the self-adjoint operator $\hat{\mathrm{h}}_0$ in the Hilbert space $\ell^{2,s}[(\mathbb{Z}^d)^2]$:

$$\hat{\mathrm{h}}_0 = \Delta \otimes I + I \otimes \Delta,$$

where $\otimes$ denotes the tensor product and I is the identity operator on $L^2(\mathbb{Z}^d)$. The Hamiltonian $\hat{\mathrm{h}}_\mu$ of a system of two identical particles with the two-particle pair zero-range attractive interaction $\mu\hat{v}$ is a bounded perturbation of the free Hamiltonian $\hat{\mathrm{h}}_0$ on the Hilbert space $\ell^{2,s}[(\mathbb{Z}^d)^2]$

$$\hat{\mathrm{h}}_\mu = \hat{\mathrm{h}}_0 + \mu\hat{v}.$$

Here $\mu < 0$ is coupling constant and

$$(\hat{v}\hat{\psi})(x_1, x_2) = \delta_{x_1x_2}\hat{\psi}(x_1, x_2), \quad \hat{\psi} \in \ell^{2,s}[(\mathbb{Z}^d)^2],$$

where $\delta_{x_1x_2}$ is the Kronecker delta.

Similarly, the free Hamiltonian $\hat{\mathrm{H}}_0$ of a system of three identical particles on the d-dimensional lattice $\mathbb{Z}^d$ with mass $m = 1$ is defined on the Hilbert space $\ell^{2,s}[(\mathbb{Z}^d)^3]$:

$$\widehat{H}_0 = \Delta \otimes I \otimes I + I \otimes \Delta \otimes I + I \otimes I \otimes \Delta.$$

The Hamiltonian $\hat{\mathrm{H}}_\mu$ of a system of three identical particles with the two-particle pair zero-range interactions $\hat{v} = \hat{v}_\alpha = \hat{v}_{\beta\gamma}, \alpha, \beta, \gamma = 1,2,3$ is a bounded perturbation of the free Hamiltonian $\hat{\mathrm{H}}_0$

$$\hat{\mathrm{H}}_\mu = \hat{\mathrm{H}}_0 + \mu\hat{\mathbb{V}},$$

where $\mathbb{V} = \sum_{\alpha=1}^{3}\hat{V}_\alpha$, $V_\alpha = \hat{V}, \alpha = 1,2,3$ is the multiplication operator on $\ell^{2,s}[(\mathbb{Z}^d)^3]$ defined by

$$(\hat{V}_\alpha\hat{\psi})(x_1, x_2, x_3) = \delta_{x_\beta x_\gamma}\hat{\psi}(x_1, x_2, x_3),$$
$$\alpha \prec \beta \prec \gamma \prec \alpha,\ \alpha, \beta, \gamma = 1,2,3,\ \hat{\psi} \in \ell^{2,s}[(\mathbb{Z}^d)^3].$$

2.1. **The momentum representation.** Let $\mathbb{T}^d = (-\pi, \pi]^d$ be the $d-$ dimensional torus and $L^{2,s}[(\mathbb{T}^d)^m] \subset L^2[(\mathbb{T}^d)^m]$ be the subspace of symmetric functions defined on the Cartesian power $(\mathbb{T}^d)^m$, $m \in \mathbb{N}$.

Let $\mathcal{F}:\ L^2(\mathbb{T}^d) \to \ell^2(\mathbb{Z}^d)$ be the standard Fourier transform

$$\mathcal{F}:\ \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d), \quad [\mathcal{F}(f)](p) := \sum_{x\in\mathbb{Z}^d} e^{-\mathrm{i}(p,x)}\, f(x)$$

with the inverse

$$\mathcal{F}^*:\ L^2(\mathbb{T}^d) \to \ell^2(\mathbb{Z}^d), \quad [\mathcal{F}^*(\psi)](x) := \int_{\mathbb{T}^d} e^{\mathrm{i}(p,x)}\,\psi(p)\,\eta(dp)$$

and $\eta(dp) = \frac{d^dp}{(2\pi)^d}$ is the (normalized) Haar measure on the torus.

It easily can be checked that Fourier transform

$$\hat{\Delta} = \mathcal{F}\Delta\,\mathcal{F}^*$$

of the Laplacian Δ is the multiplication operator by the function $\varepsilon(\cdot)$:

$$(\hat{\Delta} f)(p) = \varepsilon(p) f(p), \quad f \in L^2(\mathbb{T}^d),$$

where

$$\varepsilon(p) = \sum_{i=1}^{d} (1 - \cos p^{(i)}), \quad p = (p^{(1)}, ..., p^{(d)}) \in \mathbb{T}^d.$$

The two-particle total Hamiltonian h_μ in the momentum representation is given on $L^{2,s}[(\mathbb{T}^d)^2]$ as follows

$$\mathrm{h}_\mu = \mathrm{h}_0 + \mu v.$$

Here the free Hamiltonian h_0 is of the form

$$\mathrm{h}_0 = \hat{\Delta} \otimes \hat{I} + \hat{I} \otimes \hat{\Delta},$$

where $\hat{I}$ is the identity operator on $L^2(\mathbb{T}^d)$. It is easy to see that the operator h_0 is the multiplication operator by the function $\varepsilon(k_1) + \varepsilon(k_2)$:

$$(\mathrm{h}_0 f)(k_1, k_2) = [\varepsilon(k_1) + \varepsilon(k_2)] f(k_1, k_2), \ f \in L^{2,s}[(\mathbb{T}^d)^2]$$

and v is the convolution type integral operator

$$(vf)(k_1, k_2) = \int\limits_{(\mathbb{T}^d)^2} \delta(k_1 + k_2 - k_1' - k_2') f(k_1', k_2') \eta(dk_1') \eta(dk_2')$$

$$= \int\limits_{\mathbb{T}^d} f(k_1', k_1 + k_2 - k_1') \eta(dk_1'), \ f \in L^{2,s}[(\mathbb{T}^d)^2],$$

where $\delta(\cdot)$ is the $d-$ dimensional Dirac delta function.

The three-particle Hamiltonian in the momentum representation is given as bounded self-adjoint operator on the Hilbert space $L^{2,s}[(\mathbb{T}^d)^3]$

$$\mathrm{H}_\mu = \mathrm{H}_0 + \mu(V_1 + V_2 + V_3),$$

where H_0 is of the form

$$\mathrm{H}_0 = \hat{\Delta} \otimes \hat{I} \otimes \hat{I} + \hat{I} \otimes \hat{\Delta} \otimes \hat{I} + \hat{I} \otimes \hat{I} \otimes \hat{\Delta},$$

i.e., the free Hamiltonian H_0 is the multiplication operator by the function $\sum_{\alpha=1}^{d} \varepsilon(k_\alpha)$:

$$(\mathrm{H}_0 f)(k_1, k_2, k_3) = [\sum_{\alpha=1}^{3} \varepsilon(k_\alpha)] f(k_1, k_2, k_3),$$

and $V_\alpha = V, \alpha = 1, 2$ are convolution type integral operators

$$(V_\alpha f)(k_\alpha, k_\beta, k_\gamma)$$
$$= \int\limits_{(\mathbb{T}^d)^3} \delta(k_\alpha - k_\alpha') \, \delta(k_\beta + k_\gamma - k_\alpha' - k_\beta') f(k_\alpha', k_\beta', k_\gamma') \eta(dk_\alpha') \eta(dk_\beta') \eta(dk_\gamma')$$
$$= \int\limits_{\mathbb{T}^d} f(k_\alpha, k_\beta', k_\beta + k_\gamma - k_\beta') \eta(dk_\beta'), \quad f \in L^{2,s}[(\mathbb{T}^d)^3].$$

3. Decomposition of the energy operators into von Neumann direct integrals. Quasi-momentum and coordinate systems

Let $k = k_1 + k_2 \in \mathbb{T}^d$ resp. $K = k_1 + k_2 + k_3 \in \mathbb{T}^d$ be the *two-* resp. *three-particle quasi-momentum* and the set $\mathbb{Q}_k$ resp. $\mathbb{Q}_K$ is defined as follows

$$\mathbb{Q}_k = \{(k_1, k - k_1) \in (\mathbb{T}^d)^2 : k_1 \in \mathbb{T}^d k - k_1 \in \mathbb{T}^d\}$$

resp.

$$\mathbb{Q}_K = \{(k_1, k_2, K - k_1 - k_2) \in (\mathbb{T}^d)^3 : k_1, k_2 \in \mathbb{T}^d, K - k_1 - k_2 \in \mathbb{T}^d\}.$$

We introduce the mapping

$$\pi_1 : (\mathbb{T}^d)^2 \to \mathbb{T}^d, \quad \pi_1(k_1, k_2) = k_1$$

resp.

$$\pi_2 : (\mathbb{T}^d)^3 \to (\mathbb{T}^d)^2, \quad \pi_2(k_1, k_2, k_3) = (k_1, k_2).$$

Denote by π_k, $k \in \mathbb{T}^d$ resp. π_K , $K \in \mathbb{T}^d$ the restriction of π_1 resp. π_2 onto $\mathbb{Q}_k \subset (\mathbb{T}^d)^2$, resp. $\mathbb{Q}_K \subset (\mathbb{T}^d)^3$, that is,

$$\pi_k = \pi_1|_{\mathbb{Q}_k} \quad \text{and} \quad \pi_K = \pi_2|_{\mathbb{Q}_K}.$$

It is useful to remark that $\mathbb{Q}_k,\ k \in \mathbb{T}^d$ resp. $\mathbb{Q}_K,\ K \in \mathbb{T}^d$ are the $d-$ resp. $2d-$ dimensional manifold isomorphic to $\mathbb{T}^d$ resp. $(\mathbb{T}^d)^2$.

Lemma 3.1. *The mapping π_k, $k \in \mathbb{T}^d$ resp. π_K, $K \in \mathbb{T}^d$ is bijective from $\mathbb{Q}_k \subset (\mathbb{T}^d)^2$ resp. $\mathbb{Q}_K \subset (\mathbb{T}^d)^3$ onto $\mathbb{T}^d$ resp. $(\mathbb{T}^d)^2$ with the inverse mapping given by*

$$(\pi_k)^{-1}(k_1) = (k_1, k - k_1)$$

resp.

$$(\pi_K)^{-1}(k_1, k_2) = (k_1, k_2, K - k_1 - k_2).$$

Let $L^{2,e}(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$ be the subspace of even functions. Decomposing the Hilbert space $L^{2,s}[(\mathbb{T}^d)^2]$ resp. $L^{2,s}[(\mathbb{T}^d)^3]$ into the direct integral

$$L^{2,s}[(\mathbb{T}^d)^2] = \int_{k \in \mathbb{T}^d} \oplus L^{2,e}(\mathbb{T}^d)\eta(dk)$$

resp.

$$L^{2,s}[(\mathbb{T}^d)^3] = \int_{K \in \mathbb{T}^d} \oplus L^{2,s}[(\mathbb{T}^d)^2]\eta(dK)$$

yields the decomposition of the Hamiltonian h_μ resp. H_μ into the direct integral

$$\mathrm{h}_\mu = \int\limits_{k \in \mathbb{T}^d} \oplus \tilde{h}_\mu(k)\eta(dk)$$

resp.

$$\mathrm{H}_\mu = \int\limits_{K \in \mathbb{T}^d} \oplus \tilde{H}_\mu(K)\eta(dK).$$

3.1. **The discrete Schrödinger operators.** The fiber operator $\tilde{h}_\mu(k),\, k \in \mathbb{T}^d$ is unitarily equivalent to the operators $h_\mu(k),\, k \in \mathbb{T}^d$ acting in $L^{2,e}(\mathbb{T}^d) \subset L_2(\mathbb{T}^d)$:

$$h_\mu(k) = h_0(k) + \mu v. \tag{3.1}$$

The operator $h_0(k)$ is the multiplication operator by the function $\mathcal{E}_k(p)$:

$$(h_0(k)f)(p) = \mathcal{E}_k(p)f(p), \quad f \in L^{2,e}(\mathbb{T}^d),$$

where

$$\mathcal{E}_k(p) = \varepsilon(\frac{k}{2} - p) + \varepsilon(\frac{k}{2} + p)$$

and

$$(vf)(p) = \int_{\mathbb{T}^d} f(q)d\eta(q), \quad f \in L^{2,e}(\mathbb{T}^d).$$

The fiber operator $\tilde{H}_\mu(K),\ K \in \mathbb{T}^d$ is unitarily equivalent to the operator $H_\mu(K)$, $K \in \mathbb{T}^d$ given by

$$H_\mu(K) = H_0(K) + \mu(V_1 + V_2 + V_3), V_\alpha = V,\ \alpha = 1, 2, 3.$$

The operator $H_0(K),\ K \in \mathbb{T}^d$ acts in the Hilbert space $L^{2,s}[(\mathbb{T}^d)^2]$ and is of the form

$$(H_0(K)f)(p, q) = E(K; p, q)f(p, q), \quad f \in L^{2,s}[(\mathbb{T}^d)^2],$$

where

$$E(K; p, q) = \varepsilon(K - p - q) + \varepsilon(p) + \varepsilon(q).$$

The operator $\mathbb{V} = V_1 + V_2 + V_3$ acts in $L^{2,s}[(\mathbb{T}^d)^2]$ and in the coordinates $(p, q) \in (\mathbb{T}^d)^2$ can be written as follows

$$(\mathbb{V}f)(p, q) = \int_{\mathbb{T}^d} f(p, t)\eta(dt) + \int_{\mathbb{T}^d} f(t, q)\eta(dt) + \int_{\mathbb{T}^d} f(t, K - p - q)\eta(dt), f \in L^{2,s}[(\mathbb{T}^d)^2].$$

4. Statement of the main results

According to Weyl's theorem [15] the essential spectrum $\sigma_{\mathrm{essspec}}(h_\mu(k))$ of the operator $h_\mu(k),\ k \in \mathbb{T}^d$ coincides with the spectrum $\sigma(h_0(k))$ of $h_0(k)$. More precisely,

$$\sigma_{\mathrm{essspec}}(h_\mu(k)) = [\mathcal{E}_{\min}(k),\ \mathcal{E}_{\max}(k)],$$

where

$$\mathcal{E}_{\min}(k) \equiv \min_{p \in \mathbb{T}^d} \mathcal{E}_k(p) = 2\sum_{i=1}^{d}[1 - \cos(\frac{k^{(i)}}{2})]$$

$$\mathcal{E}_{\max}(k) \equiv \max_{p \in \mathbb{T}^d} \mathcal{E}_k(p) = 2\sum_{i=1}^{d}[1 + \cos(\frac{k^{(i)}}{2})].$$

The following Theorem states the existence of a unique eigenvalue of the operator $h_\mu(k)$.

Theorem 4.1. *For any $\mu < 0$ and $k \in \mathbb{T}^d$, $d = 1, 2$ the operator $h_\mu(k)$ has a unique eigenvalue $e_\mu(k)$, which is even on $\mathbb{T}^d$ and satisfies the relations $e_\mu(k) < \mathcal{E}_{\min}(k),\ k \in \mathbb{T}^d$ and $e_\mu(0) < e_\mu(k), k \neq 0$. Moreover, for any $\mu < 0$ the eigenvalue $e_\mu(k)$ is regular function in $k \in \mathbb{T}^d$.*

The associated eigenfunction $f_{\mu,e_\mu(k)}(\cdot), k \in \mathbb{T}^d$ *is a regular function on* $\mathbb{T}^d$ *and has the form*

$$f_{\mu,e_\mu(k)}(\cdot) = \frac{\mu c(k)}{\mathcal{E}_k(\cdot) - e_\mu(k)},$$

where $c(k) \neq 0,\ k \in \mathbb{T}^d$ *is a normalizing constant. Moreover, the vector valued mapping*

$$f_\mu : \mathbb{T}^d \to L^2[\mathbb{T}^d, \eta(dk); L^{2,e}(\mathbb{T}^d)],\ k \to f_{\mu,e_\mu(k)}$$

is regular on $\mathbb{T}^d$*.*

Theorem 4.1 can be proven in the same way as in[9].

We note that the spectrum $\sigma_{\text{spec}}(H_0(K))$ of the operator $H_0(K),\ K \in \mathbb{T}^d$ is the segment $[\mathrm{E}_{\min}(K), \mathrm{E}_{\max}(K)]$, where

$$\mathrm{E}_{\min}(K) \equiv \min_{p,q\in\mathbb{T}^d} E(K;p,q),\ \mathrm{E}_{\max}(K) \equiv \max_{p,q\in\mathbb{T}^d} E(K;p,q).$$

We describe the essential spectrum of the three-particle operator $H_\mu(K),\ K \in \mathbb{T}^d$ by the spectrum of the non perturbed operator $H_0(K),\ K \in \mathbb{T}^d$ and the discrete spectrum of the two-particle operator $h_\mu(k),\ k \in \mathbb{T}^d$ in the following theorem.

Theorem 4.2. *For any* $\mu < 0$ *the essential spectrum* $\sigma_{\text{essspec}}(H_\mu(K))$ *of* $H_\mu(K),\ K \in \mathbb{T}^d$ *is described as follows*

$$\sigma_{\text{essspec}}(H_\mu(K)) = \cup_{k\in\mathbb{T}^d}\{e_\mu(k) + \varepsilon(K-k)\} \cup [E_{\min}(K), E_{\max}(K)],$$

where $e_\mu(k)$ *is a unique eigenvalue of the operator* $h_\mu(k), k \in \mathbb{T}^d$*.*

The next theorem states the finiteness of the number of eigenvalues for the Schrödinger operator $H_\mu(K)$ and the analyticity of the eigenvalues and associated eigenfunctions.

Let $U_{\delta(K)}[p_\mu(K)] = \{K \in \mathbb{T}^d : |K - p_\mu(K)| < \delta\}$ be $\delta = \delta(K)-$ neighborhood of the point $p_\mu(K) \in \mathbb{T}^d$.

Theorem 4.3. *Let* $d = 1, 2$ *and* $\mu < 0$*. Then*

(i) *There exists* $\delta > 0$ *such that for each* $K \in U_\delta[0]$ *the operator* $H_\mu(K)$ *has finite number of eigenvalues* $E_{1,\mu}(K), ..., E_{n,\mu}(K)$ *lying below the bottom of the essential spectrum* $\sigma_{\text{essspec}}(H_\mu(K))$ *with the associated bound states*

$$\psi_{\mu,E_{1,\mu}(K)}(\cdot), ..., \psi_{\mu,E_{n,\mu}(K)}(\cdot) \in L^{2,s}[(\mathbb{T}^d)^2],\ K \in U_\delta[0].$$

(ii) *The eigenfunction* $f_{\mu,E_\mu(K)}(\cdot,\cdot) \in L^{2,s}[(\mathbb{T}^d)^2]$ *of* $H_\mu(K)$ *associated to the eigenvalue* $E_\mu(K), K \in U_\delta[0]$ *is regular in* $(p,q) \in (\mathbb{T}^d)^2$*. Moreover, each eigenvalue* $E_\mu(K), K \in U_\delta[0]$ *of* $H_\mu(K)$ *is a regular function in* $K \in U_\delta[0]$ *and the vector valued mapping*

$$f_\mu : U_\delta[0] \to L^2[U_\delta[0], \eta(dK); L^{2,s}(\mathbb{T}^d)^2],\ K \to f_{\mu,E_\mu(K)}$$

is also regular on $\mathbb{T}^d$*.*

Corollary 4.4. *The two-particle Hamiltonian* h_μ *has a unique isolated band spectrum and the three-particle Hamiltonian* H_μ *have a finite number band spectrum.*

Denote by $\tau_{\text{spec}}(H_\mu(K))$ resp. $\tau_{\text{essspec}}(H_\mu(K))$ the bottom of the spectrum resp. essential spectrum of the three-particle Schrödinger operator $H_\mu(K),\ K \in \mathbb{T}^d,$ i.e.,

$$\tau_{\text{spec}}(H_\mu(K)) = \inf_{||f||=1} (H_\mu(K)f, f).$$

resp.

$$\tau_{\mathrm{essspec}}(H_\mu(K)) = \inf \sigma_{\mathrm{essspec}}(H_\mu(K)). \tag{4.1}$$

Let

$$\sigma_{\mathrm{esstwo}}(H_\mu(K)) = \cup_{k\in\mathbb{T}^d}\{e_\mu(k) + \varepsilon(K-k)\}.$$

resp.

$$\sigma_{\mathrm{essthree}}(H_\mu(K)) = [E_{\min}(K), E_{\max}(K)]$$

be the two-particle resp. three-particle essential spectrum of $H_\mu(K),\ K\in\mathbb{T}^d$ and

$$\tau_{\mathrm{esstwo}}(H_\mu(K)) = \inf \sigma_{\mathrm{esstwo}}(H_\mu(K))$$

resp.

$$\tau_{\mathrm{essthree}}(H_\mu(K)) = E_{\min}(K) = \inf_{||f||=1}[(H_0(K)f,f)]$$

be the bottom of the two-particle resp. three-particle essential spectrum.

Remark 4.5. *For the operator $H_\mu(K)$ associated to a system of three bosons on the lattice $\mathbb{Z}^d, d=1,2$ Theorems 4.1 and 4.2 give*

$$\sigma_{\mathrm{esstwo}}(H_\mu(K)) \neq \emptyset$$

and

$$\tau_{\mathrm{esstwo}}(H_\mu(K)) < \tau_{\mathrm{essthree}}(H_\mu(K))$$

and hence

$$\sigma_{\mathrm{essthree}}(H_\mu(K)) \subset \sigma_{\mathrm{essspec}}(H_\mu(K)).$$

Consequently, the inequality

$$\tau_{\mathrm{essspec}}(H_\mu(K)) < \tau_{\mathrm{essthree}}(H_\mu(K))$$

holds, which allows to prove the finiteness of the number of bound states of three interacting bosons on the lattice $\mathbb{Z}^d, d=1,2$.

Remark 4.6. *We remark that for the three-particle Schrödinger operator $H_\mu(K)$, associated to a system of three bosons in the three-dimensional lattice $\mathbb{Z}^3$, there exists $\mu_0>0$ such that*

$$\sigma_{\mathrm{essspec}}(H_{\mu_0}(0)) = \sigma_{\mathrm{essthree}}(H_{\mu_0}(0))$$

and hence

$$\tau_{\mathrm{essspec}}(H_{\mu_0}(0)) = \tau_{\mathrm{essthree}}(H_{\mu_0}(0)).$$

At the same time for any nonzero $K\in\mathbb{T}^3$ the following relation

$$\sigma_{\mathrm{esstwo}}(H_{\mu_0}(K)) \neq \emptyset$$

holds and hence

$$\tau_{\mathrm{esstwo}}(H_{\mu_0}(K)) < \tau_{\mathrm{essthree}}(H_{\mu_0}(K))$$

and

$$\sigma_{\mathrm{essthree}}(H_{\mu_0}(K)) \subset \sigma_{\mathrm{essspec}}(H_{\mu_0}(K)).$$

Thus only the operator $H_{\mu_0}(0)$ may have an infinite number of eigenvalues below the bottom of the three-particle continuum (Efimov's effect)[3, 8]*, which yields the existence of an infinite number of bound states.*

5. THE ESSENTIAL SPECTRUM OF THE OPERATOR $H_\mu(K)$.

Since we are considering the system of identical particles, there is only one channel operator $H_{\mu,ch}(K), K \in\mathbb{T}^d,\ d=1,2$ defined in the Hilbert space $L^{2,s}[(\mathbb{T}^d)^2]=L^2(\mathbb{T}^d)\otimes L^{2,e}(\mathbb{T}^d)$ as

$$H_{\mu,ch}(K)=H_0(K)+\mu V.$$

The operators $H_0(K)$ and $V=V_\alpha$ act as follows

$$(H_0(K)f)(p,q)=\mathcal{E}(K;p,q)f(p,q),\quad f\in L^{2,s}[(\mathbb{T}^d)^2],$$

where

$$\mathcal{E}(K;p,q)=\varepsilon(K-p)+\varepsilon(\frac{p}{2}-q)+\varepsilon(\frac{p}{2}+q)$$

and

$$(Vf)(p,q)=\int\limits_{\mathbb{T}^d} f(p,t)\eta(dt),\quad f\in L^{2,s}[(\mathbb{T}^d)^2].$$

The decomposition of the space $L^{2,s}[(\mathbb{T}^d)^2]$ into the direct integral

$$L^{2,s}[(\mathbb{T}^d)^2]=\int\limits_{k\in\mathbb{T}^d}\oplus L^{2,e}(\mathbb{T}^d)\eta(dk)$$

yields for the operator $H_{\mu,ch}(K)$ the decomposition

$$H_{\mu,ch}(K)=\int\limits_{k\in\mathbb{T}^d}\oplus h_\mu(K,k)\eta(dk).$$

The fiber operator $h_\mu(K,k)$ acts in the Hilbert space $L^{2,e}(\mathbb{T}^d)$ and has the form

$$h_\mu(K,k)=h_\mu(k)+\varepsilon(K-k)I, \tag{5.1}$$

where $I=I_{L^{2,e}(\mathbb{T}^d)}$ is the identity operator and $h_\mu(k)$ is the two-particle operator defined by (3.1). The representation (5.1) of the operator $h_\mu(K,k)$ and Theorem 4.1 yield for the spectrum of operator $h_\mu(K,k)$ the equality

$$\sigma(h_\mu(K,k))=Z_\mu(K,k)\cup\left[\mathcal{E}_{\min}(k),\mathcal{E}_{\max}(k)\right], \tag{5.2}$$

where

$$Z_\mu(K,k)=e_\mu(k)+\varepsilon(K-k)$$

and $e_\mu(k)$ is the unique eigenvalue of the operator $h_\mu(k)$.

The spectrum of the channel operator $H_{\mu,ch}(K),\ K\in\mathbb{T}^d$ is described in the following

Lemma 5.1. *The equality holds*

$$\sigma(H_{\mu,ch}(K))=\cup_{k\in\mathbb{T}^d}\{Z_\mu(K,k)\}\cup[E_{\min}(K),E_{\max}(K)].$$

Proof. The theorem (see, e.g.,[15]) on the spectrum of decomposable operator and the structure (5.2) of the spectrum of $h_\mu(K,k)$ give the proof. □

The essential spectrum of $H_\mu(K),\ K\in\mathbb{T}^d$ is described in the following

Theorem 5.2. *The equality*

$$\sigma_{\text{essspec}}(H_\mu(K))=\sigma(H_{\mu,ch}(K))$$

holds.

Proof. Theorem 5.2 can be proven by the same way as Theorem 3.2 in [3]. □

Theorems 4.1 and 5.2 yield that the bottom $\tau_{\mathrm{essspec}}(H_\mu(K))$ of the essential spectrum of the operator $H_\mu(K)$ less than the bottom $\tau_{\mathrm{spec}}(H_0(K)) = \tau_{\mathrm{essthree}}(H_\mu(K))$ of the spectrum of the non-perturbed operator $H_0(K)$, which is attribute for the three-particle Schrödinger operators on the lattice $\mathbb{Z}^d$ and Euclid space $\mathbb{R}^d$ in dimensions $d = 1, 2$.

Lemma 5.3. *For any $\mu < 0$ and $K \in \mathbb{T}^d, d = 1, 2$ the bottom of the essential spectrum of $H_\mu(K)$ satisfies the relations*

$$\tau_{\mathrm{essspec}}(H_\mu(K)) < \tau_{\mathrm{essthree}}(H_\mu(K)) = \tau_{\mathrm{spec}}(H_0(K)) = E_{\min}(K)$$

holds, where $\tau_{\mathrm{essspec}}(H_\mu(K))$ is defined in (4.1).

Proof. Theorem 4.1 yields that for any $k \in \mathbb{T}^d$ the operator $h_\mu(k)$ has a unique eigenvalue $e_\mu(k) < 2\,\varepsilon(\frac{k}{2}) = \mathcal{E}_{\min}(k)$.

Hence

$$Z_\mu(K, k)\Big|_{k=\frac{2K}{3}} = e_\mu(\frac{2K}{3}) + \varepsilon(\frac{K}{3}) < 2\varepsilon(\frac{K}{3}) + \varepsilon(\frac{K}{3}) = 3\varepsilon(\frac{K}{3}) = E_{\min}(K).$$

The definition of $\tau_{\mathrm{essspec}}(H_\mu(K))$ gives

$$\begin{aligned} &\tau_{\mathrm{essspec}}(H_\mu(K)) = \tau_{\mathrm{spec}}(H^{ch}_\mu(K)) \\ &= \inf \sigma(H^{ch}_\mu(K)) = \inf_{k\in\mathbb{T}^d} Z_\mu(K, k) \le e_\mu(\frac{2K}{3}) + \varepsilon(\frac{K}{3}) < 3\varepsilon(\frac{K}{3}), \end{aligned}$$

which proves Lemma 5.3. □

6. Proof of the main results

For any $K, k \in \mathbb{T}^d, d = 1, 2$ the essential spectrum $\sigma_{\mathrm{essspec}}(h_\mu(K, k))$ of the operator $h_\mu(K, k)$, $K, k \in \mathbb{T}^d$ coincides with the spectrum $\sigma(h_0(K, k))$ of $h_0(K, k)$. More precisely,

$$\sigma_{\mathrm{essspec}}(h_\mu(K, k)) = [\mathcal{E}_{\min}(K, k),\ \mathcal{E}_{\max}(K, k)].$$

$$E_{\min}(K, k) = \min_q E(K, k\,; q) = \min_q \mathcal{E}_k(q) + \varepsilon(K - k) = 2\varepsilon(\frac{k}{2}) + \varepsilon(K - k)$$

$$E_{\max}(K, k) = \max_q E(K, k\,; q) = \max_q \mathcal{E}_k(q) + \varepsilon(K - k) = [2d - \varepsilon(\frac{k}{2})] + \varepsilon(K - k).$$

The determinant $\Delta_\mu(K, k\,; z),\ K, k \in \mathbb{T}^d, d = 1, 2$ associated to the operator $h_\mu(K, k)$ can be defined as an regular function in $\mathrm{C} \setminus [E_{\min}(K, k),\ E_{\max}(K, k)]$ as

$$\Delta_\mu(K, k\,; z) = 1 + \mu \int_{\mathbb{T}^d} \frac{\eta(dk)}{E(K, k\,; q) - z}.$$

Let $\mathrm{L}_\mu(K, z),\ K \in \mathbb{T}^d,\ z < \tau_{\mathrm{essspec}}(H_\mu(K))$ be a self-adjoint operator defined in $L^2(\mathbb{T}^d)$ as

$$[\mathrm{L}_\mu(K, z)w](p) = -\mu \int_{\mathbb{T}^d} \frac{\Delta_\mu^{-\frac{1}{2}}(K, p, z)\Delta_\mu^{-\frac{1}{2}}(K, q, z)}{E(K; p, q) - z} w(q)\eta(dq), w \in L_2(\mathbb{T}^d).$$

The operator $\mathrm{L}_\mu(K, z)$ is a lattice analogue of the Birman-Schwinger operator that has been introduced in [8] to investigate Efimov's effect for the three-particle lattice Schrödinger operator $H_\mu(K)$.

For a bounded self–adjoint operator A in a Hilbert space $\mathcal{H}$ and for each $\gamma \in \mathbb{R}$ we define the number $n_+[\gamma, A]$ resp. $n_-[\gamma, A]$ as

$n_+[\gamma, A]$:

$= \max\{\dim \mathcal{H}_A^+(\gamma) : \mathcal{H}_A^+(\gamma) \subset \mathcal{H}$ subspace with $\langle A\varphi, \varphi\rangle > \gamma,\ \varphi \in \mathcal{H}_A^+(\gamma),\ ||\varphi|| = 1\}$

resp.

$n_-[\gamma, A]$:

$= \max\{\dim \mathcal{H}_A^-(\gamma) : \mathcal{H}_A^-(\gamma) \subset \mathcal{H}$ subspace with $\langle A\varphi, \varphi\rangle < \gamma,\ \varphi \in \mathcal{H}_A^-(\gamma),\ ||\varphi|| = 1\}$.

If some point of the essential spectrum of A is greater resp. smaller than γ, then $n_+[\gamma, A]$ resp. $n_-[\gamma, A]$ is equal to infinity. If $n_+[\gamma, A]$ resp. $n_-[\gamma, A]$ is finite, then it is equal to the number of the eigenvalues (counting multiplicities) of A, which are greater resp. smaller than γ (see, for instance, Glazman lemma [13]).

Remark 6.1. *Theorem 4.2 yields that for any $K \in \mathbb{T}^d$ the operator $H_\mu(K)$ has no essential spectrum below $\tau_{\text{essspec}}(H_\mu(K))$.*

Lemma 6.2. *(The Birman-Schwinger principle). For each $\mu < 0,\ K \in \mathbb{T}^d$ and $z < \tau_{\text{essspec}}(H_\mu(K))$ the operator $\mathrm{L}_\mu(K, z)$ is compact and the equality*

$$n_-[z, H_\mu(K)] = n_+[1, \mathrm{L}_\mu(K, z)].$$

holds. Moreover for any $\mu < 0,\ K \in \mathbb{T}^d$ the operator $\mathrm{L}_\mu(K, z)$ is continuous in $z \in (-\infty, \tau_{\text{essspec}}(H_\mu(K)))$.

Proof. We first verify the equality

$$n_-[z, H_\mu(K)] = n_+[1, -3\mu R_0^{\frac{1}{2}}(K, z) V R_0^{\frac{1}{2}}(K, z)]. \tag{6.1}$$

Assume that $u \in \mathcal{H}^-_{H_\mu(K)}(z) \subset L^{2,s}[(\mathbb{T}^d)^2]$, that is, $((H_0(K) - z)u, u) < -3\mu(Vu, u)$. Then

$$(y, y) < (-3\mu R_0^{\frac{1}{2}}(K, z) V R_0^{\frac{1}{2}}(K, z) y, y), \quad y = R_0^{\frac{1}{2}}(K, z) u,$$

where $R_0(K, z)$ is the resolvent of the $H_0(K)$.Hence

$$n_-[z, H_\mu(K)] \le n_+[1, -3\mu R_0^{\frac{1}{2}}(K, z) V R_0^{\frac{1}{2}}(K, z)].$$

Reversing the argument we get the opposite inequality, which proves (6.1).

Note that any nonzero eigenvalue of $R_0^{\frac{1}{2}}(K, z) V^{\frac{1}{2}}$ is an eigenvalue for $V^{\frac{1}{2}} R_0^{\frac{1}{2}}(K, z)$ as well, of the same algebraic and geometric multiplicities. Therefore we get

$$n_+[1, -3\mu R_0^{\frac{1}{2}}(K, z) V R_0^{\frac{1}{2}}(K, z)] = n_+[1, -3\mu V^{\frac{1}{2}} R_0(K, z) V^{\frac{1}{2}}].$$

Let us check the equality

$$n_+[1, -3\mu V^{\frac{1}{2}} R_0(K, z) V^{\frac{1}{2}}] = n_+[1, \mathrm{L}_\mu(K, z)].$$

We show that for any

$$u \in \mathcal{H}^+_{-3\mu V^{\frac{1}{2}} R_0(K,z) V^{\frac{1}{2}}}(1)$$

there exists $y \in \mathcal{H}^+_{\mathrm{L}_\mu(K,z)}(1)$ such that $(y, y) < (\mathrm{L}_\mu(K, z) y, y)$.

Let $u \in \mathcal{H}^+_{-3\mu V^{\frac{1}{2}} R_0(K,z) V^{\frac{1}{2}}}(1)$. Then

$$(u, u) < -3\mu(V^{\frac{1}{2}} R_0(K, z) V^{\frac{1}{2}} u, u)$$

and

$$([I+\mu V^{\frac{1}{2}}R_0(K,z)V^{\frac{1}{2}}]u,u) < -2\mu(V^{\frac{1}{2}}R_0(K,z)V^{\frac{1}{2}}u,u). \tag{6.2}$$

Since $z<\tau_{\mathrm{essspec}}(H_\mu(K))$ the operator $I+\mu V^{\frac{1}{2}}R_0(K,z)V^{\frac{1}{2}}$ is invertible and positive the operator $W_\mu^{\frac{1}{2}}(K,z)=(I+\mu V^{\frac{1}{2}}R_0(K,z)V^{\frac{1}{2}})^{-\frac{1}{2}}$ exists. Setting

$$y=(I+\mu V^{\frac{1}{2}}R_0(K,z)V^{\frac{1}{2}})^{\frac{1}{2}}u$$

gives us

$$(y,y) < -2\mu(W_\mu^{\frac{1}{2}}(K,z)V^{\frac{1}{2}}R_0(K,z)V^{\frac{1}{2}}W_\mu^{\frac{1}{2}}(K,z)y,y).$$

Since $W_\mu^{\frac{1}{2}}(K,z)$ is the multiplication operator by the function $\Delta_\mu^{-\frac{1}{2}}(K,p\,;z)$ the inequalities

$$(y,y)\le(\mathrm{L}_\mu(K,z)y,y)$$

and

$$n_+[1,-3\mu R_0^{\frac{1}{2}}(K,z)VR_0^{\frac{1}{2}}(K,z)]\le n_+(1,\mathrm{L}_\mu(K,z))$$

hold. By the same way one checks that

$$n_+(1,\mathrm{L}_\mu(K,z))\le n_+(1,-3\mu R_0^{\frac{1}{2}}(K,z)VR_0^{\frac{1}{2}}(K,z)).$$

□

The following lemma gives the well known relation between the eigenvalues of $h_\mu(K,k)$ and zeros of the determinant $\Delta_\mu(K,k\,;z)$ [3].

Lemma 6.3. *For all $K,k\in\mathbb{T}^d$ the number $z\in \mathrm{C}\backslash[E_{\min}(K,k),\,E_{\max}(K,k)]$ is an eigenvalue of the operator $h_\mu(K,k)$ if and only if*

$$\Delta_\mu(K,k\,;z)=0.$$

The proof of Lemma 6.3 is usual and can be found [9].

Lemma 6.4. *The following assertions (i)–(iv) hold true.*

(i) *If $f\in L^{2,s}[(\mathbb{T}^d)^2]$ solves $H_\mu(K)f=zf$, $z<\tau_{\mathrm{essspec}}(H_\mu(K))$ then*

$$\psi(p)=\Delta_\mu^{\frac{1}{2}}(K,p\,;z)\varphi(p),\ \textit{where}\ \varphi(p)=\int\limits_{\mathbb{T}^d} f(p,t)\eta(dt)\in L^2(\mathbb{T}^d)$$

solves $\psi=\mathrm{L}_\mu(k,z)\psi$.

(ii) *If $\psi\in L^2(\mathbb{T}^d)$ solves $\psi=\mathrm{L}_\mu(k,z)\psi$, then*

$$f(p,q)=\frac{\mu[\varphi(p)+\varphi(q)+\varphi(K-p-q)]}{E(K;p,q)-z}\in L^{2,s}[(\mathbb{T}^d)^2],$$

where $\varphi(p)=\Delta_\mu^{-\frac{1}{2}}(K,p\,;z)\psi(p)$, solves the equation $H_\mu(K)f=zf$.

(iii) *For any $\mu<0$ the eigenvalue $E_\mu(K)<\tau_{\mathrm{essspec}}(H_\mu(K))$ of the operator $H_\mu(K)$ and the associated eigenfunction $f\in L^{2,s}[(\mathbb{T}^d)^2]$ are regular in $K\in\mathbb{T}^d$.*

Proof. (i) Let for some $K\in\mathbb{T}^d$ and $z<\tau_{\mathrm{essspec}}(H_\mu(K))$ the equation

$$(H_\mu(K)\psi)(p,q)=z\psi(p,q), \tag{6.3}$$

i.e., the equation

$$[E(K;p,q)-z]f(p,q) = -\mu\Big[\int_{\mathbb{T}^d} f(p,t)\eta(dt) + \int_{\mathbb{T}^d} f(t,q)\eta(dt) + \int_{\mathbb{T}^d} f(K-p-q,t)\eta(dt)\Big] \tag{6.4}$$

has a solution $f \in L^{2,s}[(\mathbb{T}^d)^2]$.

Denoting by

$$\varphi(p) = \int_{\mathbb{T}^d} f(p,t)\eta(dt)$$

we rewrite the equation (6.4) as follows

$$f(p,q) = -\mu\frac{\varphi(p)+\varphi(q)+\varphi(K-p-q)}{E(K;p,q)-z} \in L^{2,s}[(\mathbb{T}^d)^2], \tag{6.5}$$

which gives for $\varphi \in L^2(\mathbb{T}^d)$ the equation

$$\varphi(p) = -\mu\int_{\mathbb{T}^d} \frac{\varphi(p)+\varphi(t)+\varphi(K-p-t)}{E(K;p,t)-z}\eta(dt). \tag{6.6}$$

Since the function $E(K;p,t)$ is invariant under $K-p-t \to t$, we have

$$\varphi(p)\left[1+\mu\int_{\mathbb{T}^d}\frac{dq}{E(K;p,q)-z}\right] = 2\mu\int_{\mathbb{T}^d}\frac{\varphi(q)}{E(K;p,q)-z}\eta(dq)$$

Denoting by $\Delta_\mu^{\frac{1}{2}}(K,p\,;z)\varphi(p) = \psi(p)$ and taking into account the inequality $\Delta_\mu(K,p\,;z) \neq 0,\ z < \tau_{\text{essspec}}(H_\mu(K))$ we get the equation

$$\psi(p) = -2\mu\int_{\mathbb{T}^d}\frac{\Delta_\mu^{-\frac{1}{2}}(K,p\,;z)\Delta_\mu^{-\frac{1}{2}}(K,q\,;z)\psi(q)}{E(K;p,q)-z}\eta(dq). \tag{6.7}$$

(ii) Assume that for some $z < \tau_{\text{essspec}}(H_\mu(K))$ the function $\psi \in L^2(\mathbb{T}^d)$ is a solution of the equation (6.7).Then $\varphi(p) = \Delta^{-\frac{1}{2}}(K,p\,;z)\psi(p) \in L^2(\mathbb{T}^d)$ is a solution of the equation (6.6). Hence the function defined by (6.5) belongs $L^{2,s}[(\mathbb{T}^d)^2]$ and is a solution of the Schrödinger equation $H_\mu(K)f = zf$, i.e., f is an eigenfunction of the operator $H_\mu(K)$ associated to the eigenvalue $z < \tau_{\text{essspec}}(H_\mu(K))$.

(iii) For all $\mu < 0,\ K \in \mathbb{T}^d$ and $z < \tau_{\text{essspec}}(H_\mu(K))$ the kernel function

$$\mathrm{L}_\mu(K,z;p,q) = -2\mu\frac{\Delta_\mu^{-\frac{1}{2}}(K,p,z)\Delta_\mu^{-\frac{1}{2}}(K,q,z)}{E(K;p,q)-z}$$

of the operator $\mathrm{L}_\mu(K,z)$ is continuous in $p,q \in \mathbb{T}^d$. Hence, for any $\mu < 0$ and $K \in \mathbb{T}^d$ the Fredholm determinant $D_\mu(K,z) = \det[I - \mathrm{L}_\mu(K,z)]$ associated to $\mathrm{L}_\mu(K,z;p,q)$ is real and regular function in $z \in (-\infty, \tau_{\text{essspec}}(H_\mu(K)))$.

Lemma 6.4 and the Fredholm theorem yield that each eigenvalue $E_\mu(K) \in (-\infty, \tau_{\text{essspec}}(H_\mu(K)))$ of the operator $H_\mu(K)$ is a zero of the determinant $D_\mu(K,z)$ and vice versa. Consequently, the compactness of the torus $\mathbb{T}^d$ and implicit function theorem yield that for each $\mu < 0$ the eigenvalue $E_\mu(K)$ of $H_\mu(K)$ is a regular function in $K \in \mathbb{T}^d,\ d = 1,2$.

Since the functions $\Delta_\mu(K,p\,;E_\mu(K))$ and $E(K;p,q) - E_\mu(K)$ are regular in $K \in \mathbb{T}^d$ the solution $\psi \in L^2[\mathbb{T}^d]$ of the equation (6.7) and hence the function

φ are regular in $K \in \mathbb{T}^d$. Hence, the eigenfunction (6.5) of the operator $H_\mu(K)$ associated to eigenvalue $E_\mu(K) < \tau_{\rm essspec}(H_\mu(K))$ is also regular in $K \in \mathbb{T}^d$. Consequently, the vector valued mapping

$$f_\mu : \mathbb{T}^d \to L^2[\mathbb{T}^d, \eta(dK); L^{2,s}[(\mathbb{T}^d)^2]],\ K \to f_{\mu,K}(\cdot,\cdot)$$

is regular in $\mathbb{T}^d$.

□

Now we are going to proof the finiteness of the number $N(K, \tau_{\rm essspec}(H_\mu(K)))$ of eigenvalues of the three-particle Schrödinger operator $H_\mu(K), K \in U_\delta[0]$.

We postpone the proof of the main theorem after the following two lemmas.

Lemma 6.5. *Let $d = 1, 2$. For any $K \in U_\delta(0)$ there are positive nonzero constants C_1 and C_2 depending on K and a neighborhood $U_{\delta(K)}[p_\mu(K)]$ of the point $p_\mu(K) \in \mathbb{T}^d$ such that for all $p \in U_{\delta(K)}[p_\mu(K)]$ the following inequalities*

$$C_1|p - p_\mu(K)|^2 \leq \Delta_\mu(K, p, \tau_{\rm essspec}(H_\mu(K))) \leq C_2|p - p_\mu(K)|^2$$

hold.

Proof. We prove Lemma 6.5 for the case $d = 2$. The point $p = 0$ is the non degenerate minimum of the function $\varepsilon(p)$, i.e.,

$$\varepsilon(p) = \frac{1}{2}p^2 + O(|p|^4) \text{ as } p \to 0.$$

Since the eigenvalue $e_\mu(p)$ lying below the essential spectrum is a unique zero of the determinant $\Delta_\mu(p, z)$ associated to operator $h_\mu(p)$, simple computations gives

$$\left(\frac{\partial^2 e_\mu(0)}{\partial p^{(i)} \partial p^{(j)}}\right)^2_{i,j=1} = C\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ C > 0.$$

Analogously, the eigenvalue $Z_\mu(0, p)$ of the operator $h_\mu(0, p)$, lying below the essential spectrum, is unique zero of the determinant $\Delta_\mu(0, p, z)$ and hence the point $p = p_\mu(0) = 0 \in \mathbb{T}^2$ is non-degenerate minimum of

$$Z_\mu(0, p) := e_\mu(p) + \varepsilon(p).$$

Therefore for any $K \in U_\delta(0)$ the point $p_\mu(K) \in U_\delta(0)$ is non degenerate minimum of the function $Z_\mu(K, p)$ and the matrix

$$B(K) = \left(\frac{\partial^2 Z_\mu(K, p_\mu(K))}{\partial p^{(i)} \partial p^{(j)}}\right)^2_{i,j=1}$$

is positive definite. Hence, the eigenvalue $Z_\mu(K, p)$ has following asymptotics

$$Z_\mu(K, p) = \tau_{\rm essspec}(H_\mu(K)) + (B(K)(p - p_\mu(K)), p - p_\mu(K)) + o(|p - p_\mu(K)|^2), \tag{6.8}$$

as $|p - p_\mu(K)| \to 0$, where $\tau_{\rm essspec}(H_\mu(K)) = Z_\mu(K, p_\mu(K))$.

For any $K, p \in \mathbb{T}^d$ there exists a $\gamma = \gamma(K, p) > 0$ neighborhood $W_\gamma(Z_\mu(K, p))$ of the point $Z_\mu(K, p) \in$ C such that for all $z \in W_\gamma(Z_\mu(K, p))$ the following equality holds

$$\Delta_\mu(K, p, z) = \sum_{n=1}^{\infty} C_n(\mu, K, p)(z - Z_\mu(K, p))^n,$$

where

$$C_1(\mu, K, p) = -\mu \int\limits_{\mathbb{T}^d} \frac{dq}{[E(K;p,q) - Z_\mu(K,p)]^2} < 0.$$

From here one can concludes that for any $K \in U_\delta(0)$ there is $U_{\delta(K)}(p_\mu(K))$ and for all $p \in U_{\delta(K)}(p_\mu(K))$ the equality

(6.9)

$$\Delta_\mu(K, p, \tau_{\text{essspec}}(H_\mu(K))) = [Z_\mu(K,p) - \tau_{\text{essspec}}(H_\mu(K))]\hat{\Delta}_\mu(K, p, \tau_{\text{essspec}}(H_\mu(K)))$$

holds, where $\hat{\Delta}_\mu(K, p_\mu(K), \tau_{\text{essspec}}(H_\mu(K)) \neq 0$. Putting (6.9) into (6.8) proves Lemma 6.5. □

Lemma 6.6. *Let $K \in U_\delta(0)$. The operator $\mathrm{L}_\mu(K,z)$ can be represented as sum of the two operators*

$$\mathrm{L}_\mu(K,z) = \mathrm{L}_\mu^{(1)}(K,z) + \mathrm{L}_\mu^{(2)}(K,z),$$

where the operator $\mathrm{L}_\mu^{(1)}(K,z), z < \tau_{\text{essspec}}(H_\mu(K)$ has finite rank and $\mathrm{L}_\mu^{(2)}(K,z), z \leq \tau_{\text{essspec}}(H_\mu(K)$ belongs to the Hilbert-Schmidt class.

Proof. We represent the operator $\mathrm{L}_\mu(K,z)$ as sum of two operators

$$\mathrm{L}_\mu(K,z) = \mathrm{L}_\mu^{(1)}(K,z) + \mathrm{L}_\mu^{(2)}(K,z),$$

where

$$[\mathrm{L}_\mu^{(1)}(K,z)w](p) = 2\mu \int\limits_{\mathbb{T}^d} \frac{\mathrm{L}_\mu^{(1)}(K,z;p,q)w(q)\eta(dq)}{\Delta_\mu^{\frac{1}{2}}(K,p,z)\Delta_\mu^{\frac{1}{2}}(K,q,z)},$$

is the finite rank operator, where

$$\mathrm{L}_\mu^{(1)}(K,z;p,q) = \frac{1}{E(K;p,p_\mu(K)) - z} + \frac{1}{E(K;p_\mu(K),q) - z} - \frac{1}{E(K;p_\mu(K),p_\mu(K)) - z}$$

and

$$[\mathrm{L}_\mu^{(2)}(K,z)w](p) = 2\mu \int\limits_{\mathbb{T}^d} \frac{\mathrm{L}_\mu^{(2)}(K,z;p,q)w(q)\eta(dq)}{\Delta_\mu^{\frac{1}{2}}(K,p,z\Delta_\mu^{\frac{1}{2}}(K,q,z)},$$

where

$$\mathrm{L}_\mu^{(2)}(K,z;p,q) = \frac{1}{E(K;p,q) - z} - \mathrm{L}_\mu^{(1)}(K,z;p,q).$$

For any $z < \tau_{\text{essspec}}(H_\mu(K))$ the kernel $\mathrm{L}_\mu^{(2)}(K,z;p,q)$ of the operator $\mathrm{L}_\mu^{(2)}(K,z)$ is a regular function at the point $(p_\mu(K), p_\mu(K))$ and $\mathrm{L}_\mu^{(2)}(K,z;p_\mu(K),p_\mu(K)) = 0$.

Lemma 5.3 yields the inequality

$$E(K;p,q) - \tau_{\text{essspec}}(H_\mu(K)) \geq E_{\min}(K) - \tau_{\text{essspec}}(H_\mu(K)) > 0.$$

Hence the functions

$$[E(K;p,q) - \tau_{\text{essspec}}(H_\mu(K))]^{-1} - \mathrm{L}_\mu^{(1)}(K, \tau_{\text{essspec}}(H_\mu(K)); p, q)$$

is regular in $(p,q) \in \mathbb{T}^2$.

So, the operator $\mathrm{L}_\mu^{(2)}(K, \tau_{\text{essspec}}(H_\mu(K))), K \in U_\delta(0)$ belongs to the Hilbert-Schmidt class. □

Proof of Theorem 4.3. For any compact operators $A_1,\ A_2$ and positive numbers $\lambda_1,\ \lambda_2$ Weyl's inequality

$$n_+[\lambda_1+\lambda_2, A_1+A_2] \le n_+[\lambda_1, A_1]+n_+[\lambda_2, A_2]$$

gives that for all $z<\tau_{\rm essspec}(H_\mu(K)),\ K\in U_\delta(0)$ the relations

$$n_+[1, \mathrm{L}_\mu(K,z)] \le n_+[\frac{1}{3}, \mathrm{L}_\mu^{(1)}(K,z)]+n_+[\frac{2}{3}, \mathrm{L}_\mu^{(2)}(K,z]) \le$$

$$\le n_+[\frac{1}{3}, \mathrm{L}_\mu^{(1)}(K,z)]+n_+[\frac{1}{3}, \mathrm{L}_\mu^{(2)}(K,\tau_{\rm essspec}(H_\mu(K)))]+$$

$$+n_+[\frac{1}{3}, \mathrm{L}_\mu^{(2)}(K,z)-\mathrm{L}_\mu^{(2)}(K,\tau_{\rm essspec}(H_\mu(K))].$$

hold. The compactness of $\mathrm{L}_\mu^{(2)}(K,\tau_{\rm essspec}(H_\mu(K))), K\in U_\delta(0)$ yields

$$n_+[\frac{1}{3}, \mathrm{L}_\mu^{(2)}(K,\tau_{\rm essspec}(H_\mu(K)))]<\infty.$$

The operator $\mathrm{L}_\mu^{(2)}(K,z)$ is continuous in $z\le\tau_{\rm essspec}(H_\mu(K))$ and converges to $\mathrm{L}_\mu^{(2)}(K,\tau_{\rm essspec}(H_\mu(K)))$ in uniform operator topology. Therefore for all sufficiently small $|\tau_{\rm essspec}(H_\mu(K))-z|$ we have

$$n_+[\frac{1}{3}, \mathrm{L}_\mu^{(2)}(K,z)-\mathrm{L}_\mu^{(2)}(K,\tau_{\rm essspec}(H_\mu(K)))]=0.$$

The dimension of rank of $\mathrm{L}_\mu^{(1)}(K,z), z<\tau_{\rm essspec}(H_\mu(K))$ does not depend on $z<\tau_{\rm essspec}(H_\mu(K))$ and

$$n_+[\frac{1}{3}, \mathrm{L}_\mu^{(1)}(K,z)]<\infty.$$

Lemma 6.2 yields that the operator $\mathrm{L}_\mu(K,\tau_{\rm essspec}(H_\mu(K))), K\in U_\delta(0)$ has a finite number eigenvalues greater than 1 and consequently by Lemma 6.2 the operator $H_\mu(K), K\in U_\delta(0)$ has a finite number of eigenvalues in the interval $(-\infty,\tau_{\rm essspec}(H_\mu(K)))$.

From Lemma 6.6 one concludes that if $E_\mu(K)<\tau_{\rm essspec}(H_\mu(K))$ is an eigenvalue of the operator $H_\mu(K)$ then the associated eigenfunction is of the form (6.5).

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